Area preserving diffeomorphisms and Yang-Mills theory in two noncommutative dimensions *

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We present some evidence that noncommutative Yang-Mills theory in two dimensions is not invariant under area preserving diffeomorphisms, at variance with the commutative case. Still, invariance under *linear* unimodular maps survives, as is proven by means of a fairly general argument.

1. Introduction

Invariance under area preserving diffeomorphisms (APD) [1] is a basic symmetry of ordinary Yang-Mills theories in two dimensions (YM_2). Thanks to this property, the theory acquires an almost topological flavor [2] and, as a consequence, can be solved. Beautiful group-theoretic methods [3] can be used to obtain exact expressions for the partition function and Wilson loop averages. In particular, explicit solutions of the Migdal-Makeenko equation [4] were obtained in two dimensions [5] for the expectation values of multiply-intersecting Wilson loops, due to the circumstance that they depend only on the areas of the windows singled out on the manifold.

The invariance under APD was believed to persist in Yang-Mills theories defined on a noncommutative two-dimensional manifold; actually it was believed to play a crucial role in the large gauge group, characteristic of noncommutative theories, which merges internal and space-time transformations. If this were the case one might expect to be able to solve noncommutative YM_2 by suitably generalizing the powerful geometric procedures employed in the ordinary commutative case.

This was suggested by an intriguing circumstance which occurs when studying the theory on a noncommutative torus. Here one can exploit Morita equivalence in order to relate the model to its dual on a commutative torus [6] where the APD invariance is granted. The theory on the noncommutative plane is then reached by a suitable limit procedure and one would expect invariance to be preserved there [7].

Wilson loop perturbative expansions in the coupling constant and in $1/\theta$, θ being the noncommutativity parameter, were performed directly on the noncommutative plane in [8,9]. All the results obtained there, at the orders checked, were consistent with APD invariance, the expressions depending solely on the area.

Only recently, the authors of [10] were able to extend to a higher order (the θ^{-2} term at $\mathcal{O}(g^4)$) those results, and found different answers for a Wilson loop on a circle and on a rectangle of the same area. Their result motivated the system-

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atic investigation presented in [11], where Wilson loops based on a wide class of contours with the same area are considered. The main issue of [11] is that, indeed, invariance under APD is lost even for smooth contours, the breaking being rooted in the non-local nature of the Moyal product. Still a residual symmetry survives, precisely the invariance under *linear* unimodular transformations (SL(2,$\mathbf{R}$)).

This report is devoted to illustrate the main points of [11], which the reader is invited to consult for details and further references.

Perturbative evaluations of a Wilson loop, according to a well established procedure [12], are most easily performed, in two euclidean dimensions, by choosing an axial gauge $n_\mu A_\mu = 0$, n_μ being the (constant) gauge vector. As a matter of fact in such a gauge the self-interaction terms are missing and no Faddeev-Popov determinant is required. Gauge invariance allows to transform the gauge vector n_μ by a *linear* unimodular transformation $S_{\mu\nu}$, $detS = 1$, with real entries, namely by an element of SL(2,$\mathbf{R}$). In turn this gauge transformation can be traded for a corresponding *linear* area preserving deformation of the loop contour, as shown in [11]. In a noncommutative setting these transformations belong to the $U(\infty)$ gauge invariance group. The proof cannot be generalized to non-linear deformations, as they would require a non-constant gauge vector, which would in turn conflict with the Moyal product.

Since the considerations above seem to depend heavily on the axial gauge choice and gauge invariance is explicitly *assumed*, it looks interesting to prove the invariance under *linear* deformations of the contour *without changing the gauge vector*. This has been done for the θ^{-2} term in the expansion in $\frac{1}{\theta}$ of the Wilson loop at $\mathcal{O}(g^4)$: it was indeed the term considered by the authors in [10], where a difference was first noticed between Wilson loops of the same area but with different contours (a circular and a rectangular one). This quantity will in turn be used to prove the breaking of the *local* unimodular invariance in the noncommutative context.

The θ^{-2} term of the Wilson loop, at the perturbative order g^4, will be indicated as $W[C]$. Invariance would imply, for non self-intersecting contours, that $W[C] = kA_C^4$, where A_C is the area enclosed by the contour C, the constant k being universal, *i.e.* independent of the shape of C.

In the axial gauge $nA = 0$, the non-planar contribution to the quantum average of the Wilson loop, at the lowest relevant perturbative order $\mathcal{O}(g^4)$, can be computed for a given contour C parametrized by $z(s)\,, s \in [0,1]$

$$\mathcal{W}_4^{np} = \int [ds]\, \tilde{n}\dot{z}_1 \dots \tilde{n}\dot{z}_4 \int \frac{d^2p\, d^2q}{(2\pi)^4} \times \tag{1}$$
$$\times \frac{\exp i\{p\wedge q + p(z_1-z_3) + q(z_2-z_4)\}}{(np)^2\,(nq)^2}\,,$$

where $\tilde{n}$ is a unit vector orthogonal to n, and $p\wedge q$ is a shorthand for the antisymmetric bilinear in the momenta $\theta p_\mu \epsilon_{\mu\nu} q_\nu$; two propagators correspond to four vertices on the Wilson line; and the peculiar phase factor $e^{ip\wedge q}$, which originates from the noncommutative product, singles out the non-planar contribution. Performing the Fourier transforms in Eq. (1) leads to complicate expressions: in order to study its physical content a further expansion, in powers of $1/\theta$, can be considered. While the first terms turn out to depend only on the area A_C, the $1/\theta^2$ term is more involved and reads

$$W[C] = \frac{g^4}{4!4\pi^2\theta^2}\mathbf{P}\int\!\!\int\!\!\int\!\!\int dx_2 dy_2 dz_2 dt_2 \times \tag{2}$$
$$\times \frac{((x_1-z_1)(y_2-t_2)-(y_1-t_1)(x_2-z_2))^4}{(x_2-y_2)^2(y_2-t_2)^2}\,,$$

where the gauge $A_1 = 0$ has been chosen, the subscripts refer to the euclidean components of the coordinates, and the integral is ordered according to $x < y < z < t$. Eq. (2) can be seen [11] to be equivalent to

$$W[C] = \frac{g^4}{4!4\pi^2\theta^2} \times \big[A_C^4 +$$
$$+30\,\mathbf{P}\int\!\!\int\!\!\int x_1y_1z_1(x_1(y_2-z_2)+y_1(z_2-x_2)+$$
$$+z_1(x_2-y_2))\,dx_2\,dy_2\,dz_2 +$$
$$+\frac{5}{2}\oint\oint\left(\frac{4}{3}x_1^3y_1 + x_1^2y_1^2 + \frac{4}{3}x_1y_1^3\right)\times$$
$$\times(x_2-y_2)^2\,dx_2\,dy_2\ \big]\,, \tag{3}$$

which is more convenient for our (both analytical and numerical) computations. Here the triple integral is ordered according to $x < y < z$, while the double integral is not ordered. If we define

$$W[C] = \frac{g^4 A_C^4}{4! 4\pi^2\theta^2}\,\mathcal{I}[C]\,, \tag{4}$$

it is apparent from dimensional analysis that $\mathcal{I}[C]$ is dimensionless and characterizes the *shape* (and, *a priori*, the orientation) of a given contour.

Invariance of the quantum average of a Wilson loop under translations is automatic in noncommutative theories owing to the trace integration over space-time (see also [13]). In [11] the invariance of Eq. (3) under linear area preserving maps (elements of SL(2,$\mathbf{R}$)) is explicitly proven.

A basis for the infinitesimal generators of APD's is given by the set of vector fields

$$\{V_{m,n} \equiv n x_1^m x_2^{n-1}\partial_{x_1} - m x_1^{m-1} x_2^n \partial_{x_2}\,; \tag{5}$$
$$(m,n) \in \mathbf{N}\times\mathbf{N} - (0,0)\}\,,$$

which form an infinite dimensional Lie algebra with commutation relations

$$[V_{m,n}, V_{p,q}] = (np - mq) V_{m+p-1,n+q-1}\,. \tag{6}$$

It follows that generators with $m + n \leq 2$ span a finite subalgebra, corresponding to translations and linear unimodular maps. In [11] it is explicitely shown that Eq. (3) is left invariant by this subalgebra.

In turn the breaking of the invariance under *local*, non-linear area preserving maps is explicitly shown, at the perturbative level, for several different contours.

The computation of the Wilson loop is in principle straightforward for polygonal contours, since only polynomial integrations are required; nonetheless, a considerable amount of algebra makes it rather involved.

Here we summarize our results:

- **Triangle:** $\mathcal{I}$[Triangle] was computed for an arbitrary triangle, the result being $\mathcal{I}[\text{Triangle}] = \frac{8}{3} \simeq 2.6667$. This is consistent with SL(2,$\mathbf{R}$) invariance, since any two given triangles of equal area can be mapped into each other via a linear unimodular map.

- **Parallelogram:** $\mathcal{I}$[Parallelogram] was computed for an arbitrary parallelogram with a basis along x_1, the result being $\mathcal{I}[\text{Parallelogram}] = \frac{91}{36} \simeq 2.5278$. Again, this is consistent with SL(2,$\mathbf{R}$) invariance, by the same token as above.

- **Trapezoid:** Here we can see analytically an instance of the broken invariance. Trapezoids of equal area cannot in general be mapped into one other by a linear transformation, since the ratio of the two basis b_1/b_2 is a SL(2,$\mathbf{R}$) invariant. One might say that the space of trapezoids of a given area, modulo SL(2,$\mathbf{R}$), has at least (and indeed, exactly) one modulus, which can be conveniently chosen as the ratio b_1/b_2. Actually, the result we obtained for $\mathcal{I}$[Trapezoid] reads

$$\mathcal{I}[\text{Trapezoid}] = \tag{7}$$
$$= \frac{4(6b_1^4 + 24b_1^3 b_2 + 31 b_1^2 b_2^2 + 24 b_1 b_2^3 + 6 b_2^4)}{9(b_1+b_2)^4}\,,$$

  namely a function of b_1/b_2 only, duly invariant under the exchange of b_1 and b_2, and correctly reproducing $\mathcal{I}$[Parallelogram] when $b_1 = b_2$, and $\mathcal{I}$[Triangle] when $b_1 = 0$. It is plotted in Fig. (1) .

Thus, the main outcome of our computations is that different polygons turn out to produce different results, unless they can be mapped into each other through linear unimodular maps.

As opposed to polygonal contours, smooth contours cannot be in general computed analytically, with the noteworthy exceptions of the circle and the ellipse. A circle can be mapped to any ellipse of equal area by the forementioned area preserving linear maps. They share the result

$$\mathcal{I}[\text{Circle}] = \mathcal{I}[\text{Ellipse}] = 1 + \frac{175}{12\pi^2}\,. \tag{8}$$

In the lack of explicit computations for smooth contours, different from circles and ellipses, this scenario might have left open the question whether in the noncommutative case invariance would still be there for smooth contours of equal

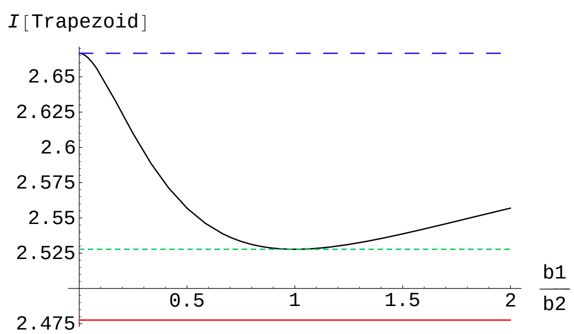

Figure 1. $\mathcal{I}$[Trapezoid] as a function of the ratio of the basis b_1/b_2; the continuous, dashed and dotted straight lines refer to $\mathcal{I}$[Circle], $\mathcal{I}$[Triangle] and $\mathcal{I}$[Square], respectively.

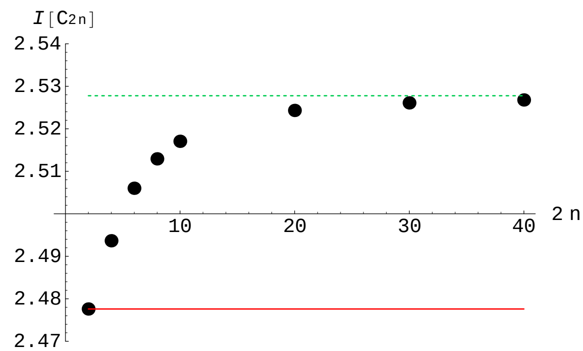

Figure 2. $\mathcal{I}[C_{2n}]$ for Fermat curves with different n; the continuous, dotted lines refer to $\mathcal{I}$[Circle], $\mathcal{I}$[Square], respectively.

area, only failing for polygons due to the presence of cusps. We did the check by numeric computations for the (even order) Fermat curves $C_{2n} \equiv \{(x,y) : \ x^{2n} + y^{2n} \ = 1\,,\ \text{with}\ n \geq 1\}$, which constitute a family of closed and smooth contours, "interpolating", in a discrete sense, between two analytically known results, *i.e.* the circle ($n = 1$) and the square (the $n \to \infty$ limit).

$\mathcal{I}[C_{2n}]$ definitely varies with n and in the $n \to \infty$ limit approaches $\mathcal{I}$[Square]. Numerical results are given in Tab. (1) and plotted in Fig. 2. Thus we conclude that invariance under area preserving diffeomorphisms does *not* hold (see also [14]).

2. Beyond the perturbation theory

Let us provide now a somewhat general argument concerning invariance under area preserving diffeomorphisms. We discuss for simplicity the $U(1)$ case.

In ordinary YM$_2$ the quantum average of a Wilson loop reads

$$W[C] = \int \mathcal{D}A\, e^{-S[A]} w[C,A]\,, \tag{9}$$

where $S[A] = \frac{1}{4}\int F_{ij}F_{kl}\eta^{ik}\eta^{jl}d^2x$ is a functional of the vector field A, and $w[C,A] = P\exp i\int_C A_i dx^i$ is a functional of A and of the contour C. This has been formulated in cartesian coordinates with metric η_{ij}.

Under a different choice of coordinates $x' = x'(x)$, $W[C]$ can be rewritten as

$$W[C] = \int \mathcal{D}A\, e^{-S_{gen}[A',g']} w[C',A'], \tag{10}$$

where $S_{gen}[A,g] = \frac{1}{4}\int F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{\det g}\, d^2x$, provided A and η transform to A', g' like tensors. Notice that $\det g$ is positive and the definition of $F_{\mu\nu}$ is left unchanged in the covariantized formulation.

We can consider now the same functional computed for the deformed contour C'

$$W[C'] = \int \mathcal{D}A\, e^{-S[A]} w[C',A], \tag{11}$$

the deformation being described by the same map $x' = x'(x)$ as above.

The condition

$$S_{gen}[A,g'] = S[A] \tag{12}$$

would describe a symmetry of the classical action. In $d = 2$, due to the circumstance that $F_{\mu\nu}$ is a two-form, we get

$$S[A] = \frac{1}{2}\int F_{12}^2 d^2x, \tag{13}$$

Table 1
Numerical results for $\mathcal{I}[C_{2n}]$ for Fermat curves with different n.

n	1	2	3	4	5	10	15	20
$\mathcal{I}[C_{2n}]$	2.4776	2.4937	2.5060	2.5129	2.5171	2.5243	2.5261	2.5268

while in $S_{gen}[A,g]$ the contractions with the inverse metric contribute a factor $(\det g)^{-1}$

$$S_{gen}[A,g] = \frac{1}{2}\int F_{12}^2 \frac{1}{\sqrt{\det g}} d^2x. \tag{14}$$

The condition Eq. (12) then amounts to $\det g = 1$, which is ensured if the Jacobian of the map is one, namely if C can be deformed to C' by an area preserving map.

The classical symmetry persists at the quantum level if $\mathcal{D}A = \mathcal{D}A'$, apart from an overall normalization; this is suggested by the circumstance that the functional jacobian turns out to be independent of the fields.

In the noncommutative theory, the expectation value of the Wilson loop becomes

$$W_{nc}[C,*] = \int \mathcal{D}A\, e^{-S_{nc}[A,*]} w_{nc}[C,A,*]\,, \tag{15}$$

where the Moyal product

$$a * b \equiv \left[\exp\,[i\frac{\theta}{2}\epsilon^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}]a(x_1)b(x_2)\right]_{|x_1=x_2} \tag{16}$$

has been introduced, and the involved functionals are defined as

$$S_{nc}[A,*] = \frac{1}{4}\int F_{ij} * F_{kl}\,\eta^{ik}\eta^{jl}\,d^2x \tag{17}$$

$$w_{nc}[C,A,*] = P_* \exp i \int_C A_i dx^i\,,$$

in which the dependence on $*$ is explicitly exhibited.

W_{nc} can be rewritten in general coordinates, provided a covariantized $*^g$ product is defined as

$$a *^g b \equiv \tag{18}$$
$$\left[\exp\,[i\frac{\theta}{2}\frac{\epsilon^{\mu\nu}}{\sqrt{\det g}}\mathcal{D}_\mu^{x_1}\mathcal{D}_\nu^{x_2}]a(x_1)b(x_2)\right]_{|x_1=x_2},$$

$\mathcal{D}_\mu$ being the covariant derivative associated to the Riemannian connection for the metric g. [4]

[4]We stress that, by introducing $*^g$, we are not formulating the theory on a curved space. Instead, we are just rewriting the theory on the flat space in general coordinates. It should be evident, from general covariance of the tensorial quantities involved, that Eq. (18) in Cartesian coordinates reproduces the usual Moyal product Eq. (16). Notice also that, since by definition $\mathcal{D}_\mu g_{\rho\sigma}(x) = 0$, it is irrelevant to choose either x_1 or x_2 as argument of $\det g(x)$ in Eq. (18), and, by the same token, $*^g$ is uneffective when acting on the metric tensor g.

Under the same reparametrization we then obtain

$$W_{nc}[C,*] = \tag{19}$$
$$\int \mathcal{D}A\, e^{-S_{nc,gen}[A',g',*^{g'}]} w_{nc}[C',A',*^{g'}],$$

where the noncommutative action in general coordinates

$$S_{nc,gen}[A,g,*^g] = \tag{20}$$
$$\frac{1}{4}\int F_{\mu\nu} *^g F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}\sqrt{\det g}\, d^2x$$

and Wilson loop

$$w_{nc,gen}[C,A,*^g] = P_{*^g} \exp i \int A_\mu dx^\mu \tag{21}$$

have been introduced.

Assuming the absence of functional anomalies also in the noncommutative case, we compare Eq. (19) with the functional $W_{nc}[C',*]$ computed for the deformed contour C'

$$W_{nc}[C',*] = \int \mathcal{D}A\, e^{-S_{nc}[A,*]} w_{nc}[C',A,*]. \tag{22}$$

The two quantities coincide if the following two sufficient conditions are met

- $*^g = *$,
- $S_{nc,gen}[A,g',*] = S_{nc}[A,*]$.

These conditions imply that the map is at most linear, since the Riemannian connection must vanish, and that its Jacobian equals unity. In conclusion, only SL(2,$\mathbf{R}$) linear maps are allowed.

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